Stochastic Programming: A tutorial – part II

DORS Tutorials 14/02/2023

Giovanni Pantuso

Department of Mathematical Sciences
University of Copenhagen
Copenhagen, Denmark
gp@math.ku.dk

Table of Contents

Applicability

Two-stage linear stochastic programs with recourse where

- ξ is a discrete random variable,
- $\mathcal{X} = \mathbb{R}_+^{n_1}$,
- $\mathcal{Y} = \mathbb{R}_+^{n_2}$.

The integer case requires some adjustments.

Recall

The deterministic equivalent problem

$$\min z = c^T x + Q(x)$$
$$\text{s.t.} Ax = b$$
$$x \geq 0$$

where

$$Q(x) = \sum_{s=1}^{S} \pi_s Q(x, \xi_s)$$

and

$$Q(x, \xi_s) = \min_y \{q_s^T y | W_s y = h_s - T_s x, y \geq 0\}.$$

Recall

$$\mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$$

Recall

$$\mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$$

$$\mathcal{K}_2(\xi_s) = \{x | \exists y \geq 0, \text{s.t.} W_s y = h_s - T_s x\}$$

Recall

$$\mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$$

$$\mathcal{K}_2(\xi_s) = \{x | \exists y \geq 0, \text{s.t.} W_s y = h_s - T_s x\}$$

$$\mathcal{K}_2 = \bigcap_{\xi \in \Xi} \mathcal{K}_2(\xi)$$

Recall

$$\mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$$

$$\mathcal{K}_2(\xi_s) = \{x | \exists y \geq 0, \text{s.t.} W_s y = h_s - T_s x\}$$

$$\mathcal{K}_2 = \bigcap_{\xi \in \Xi} \mathcal{K}_2(\xi)$$

- $\mathcal{K}_2$ is a closed and convex polyhedron
- $Q(x)$ is piecewise linear and convex in x

This will help..

A reformulation of the DEP

$$\min z = c^T x + Q(x)$$
$$\text{s.t.} x \in \mathcal{K}_1 \cap \mathcal{K}_2$$

A reformulation of the DEP

If we introduce a variable ϕ we can obtain another reformulation

$$\begin{aligned} \min z &= c^T x + \phi \\ \text{s.t.} x &\in \mathcal{K}_1 \\ x &\in \mathcal{K}_2 \\ \phi &\geq Q(x) \end{aligned}$$

A reformulation of the DEP

Polyhedral formulation, but with way too many constraints..

Idea! Drop $x \in \mathcal{K}_2$ and $\phi \geq Q(x)$ and reconstruct them iteratively... (We may not need all their constraints).

The Master Problem

At a generic iteration..

$$
\begin{aligned}
\min z &= c^T x + \phi \\
\text{s.t.} x &\in \mathcal{K}_1 \\
f_i(x) &\leq 0 \qquad i = 1, \ldots, I, \\
g_j(x, \phi) &\leq 0 \qquad j = 1, \ldots, J
\end{aligned}
$$

The Master Problem

At a generic iteration..

$$\begin{aligned}\min z = c^T x + \phi & \\ \text{s.t.} x \in \mathcal{K}_1 & \\ f_i(x) \le 0 & \qquad i = 1, \ldots, I, \\ g_j(x, \phi) \le 0 & \qquad j = 1, \ldots, J\end{aligned}$$

Initially $I = J = 0$.

Table of Contents

Feasibility

At iteration v we solve MP and find (x^v, ϕ^v).

Does $x^v \in \mathcal{K}_2$*?* Let's check:

For each s we solve the *feasibility subproblem*.

Feasibility

$$F^P(x^v, \xi_s) = \min_{y, v^+, v^-} e^\top v^+ + e^\top v^-$$
$$\text{s.t.}\; W_s y + I v^+ - I v^- = h_s - T_s x^v,$$
$$y, v^+, v^- \geq 0$$

where $e^\top = (1, \ldots, 1)$ and I is the identity matrix.

Feasibility

$$F^P(x^v, \xi_s) = \min_{y, v^+, v^-} e^\top v^+ + e^\top v^-$$
$$\text{s.t.}\, W_s y + I v^+ - I v^- = h_s - T_s x^v,$$
$$y, v^+, v^- \geq 0$$

where $e^\top = (1, \ldots, 1)$ and I is the identity matrix.

Find the differences:

$$Q(x^v, \xi_s) = \min_y \{q_s^T y | W_s y = h_s - T_s x^v, y \geq 0\}.$$

Feasibility

$$F^P(x^v,\xi_s) = \min_{y,v^+,v^-}\{e^\top v^+ + e^\top v^- | W_s y + I v^+ - I v^- = h_s - T_s x^v, y, v^+, v^- \geq 0\}$$

Its dual

$$F^D(x^v,\xi_s) = \max_{\sigma}\{\sigma^\top(h_s - T_s x^v) | \sigma^\top W_s \leq 0, \sigma^\top I \leq e^\top, -\sigma^\top I \leq e^\top\}$$

Both are always feasible. Strong duality $F^D(x^v,\xi_s) = F^P(x^v,\xi_s)$.

Feasibility

If $F^P(x^v, \xi_s) = F^D(x^v, \xi_s) = 0$ for all s then $x^v \in \mathcal{K}_2$ otherwise it does not.

Feasibility

If $F^P(x^v, \xi_s) = F^D(x^v, \xi_s) = 0$ for all s then $x^v \in \mathcal{K}_2$ otherwise it does not.

If $x^v \notin \mathcal{K}_2$ we need to tell MP that x^v is not a good solution and must be cut off.

Feasibility

Consider solution x^v to MP. If $F^D(x^v, \xi_s) > 0$ for some s, let σ_s^v be its optimal solution. Then, the inequality

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

is violated by $x^v \notin \mathcal{K}_2$.

Proof

Feasibility

Adding inequality

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

to MP will cut off solution x^v at the next iteration. We call it a *feasibility cut*.

Feasibility

Solution $x^l \in \mathcal{K}_2$ satisfies feasibility cuts

$$(\sigma_s^v)^\top (h_s - T_s x) \le 0$$

Proof

Feasibility

Summary:

- we know how verify $x^v \in \mathcal{K}_2$,
- we know that $(\sigma_s^v)^\top (h_s - T_s x) \leq 0$ will cut off infeasible solution $x^v \notin \mathcal{K}_2$,
- we know that $(\sigma_s^v)^\top (h_s - T_s x) \leq 0$ will not cut off feasible solutions.

Table of Contents

Optimality

Assume (x^v, ϕ^v) is now such that

$$x^v \in \mathcal{K}_2$$

We should now verify whether

$$\phi^v \geq \mathcal{Q}(x^v)$$

We need to calculate

$$\mathcal{Q}(x^v) = \sum_{s=1}^{S} \pi_s Q(x^v, \xi_s)$$

Optimality

For $s = 1, \ldots, S$ solve

$$Q^P(x^v, \xi_s) = \min_y \{q_s^\top y | W_s y = h_s - T_s x^v, y \geq 0\}$$

or its dual

$$Q^D(x^v, \xi_s) = \max_\rho \{\rho^\top (h_s - T_s x^v) | \rho^\top W_s \leq q_s^\top\}$$

Optimality

Observe:

- $Q^P(x^v, \xi_s)$ is feasible (and, we assume, bounded)
- $Q^P(x^v, \xi_s) = Q^D(x^v, \xi_s)$,
- $Q(x^v) = \sum_{s=1}^{S} \pi_s Q^P(x^v, \xi_s) = \sum_{s=1}^{S} \pi_s Q^D(x^v, \xi_s)$.

Optimality

If $\phi^v < Q(x^v)$, then (x^v, ϕ^v) violates

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

where ρ_s^v is the optimal solution to $Q^D(x^v, \xi_s)$. Proof

Optimality

(x^l, ϕ^l), such that $\phi^l \geq Q(x^l)$, satisfies

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

Proof

Optimality

Summarizing:

- We know how to check optimality,
- We know how to cut off (x^v, ϕ^v) such that $\phi^v < Q(x^v)$,
- We know that optimality cuts preserve (x^l, ϕ^l) such that $\phi^l \geq Q(x^l)$.

Table of Contents

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \dots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.
4. For $s = 1, \ldots, S$ solve $Q^D(x^v, \xi_s)$ and calculate $Q(x^v)$

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.
4. For $s = 1, \ldots, S$ solve $Q^D(x^v, \xi_s)$ and calculate $Q(x^v)$
5. If $\phi^v \geq Q(x^v)$, STOP (x^v, ϕ^v) is optimal otherwise add an optimality cut and return to STEP 1.

A finite algorithm

The algorithm converges

- finitely many possible cuts
- if (at most) all cuts are available, the solution to MP is optimal.

Bounds

$$c^\top x^v + \phi^v \leq z^* \leq c^\top x^v + Q(x^v)$$

Table of Contents

Dealing with integers

Integer variables in the first stage

VS

Integer variables in the second stage

Dealing with integers

Integer variables in the first stage:

Embed the L-Shaped Method into Branch and Bound.

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \quad \forall x$

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \quad \forall x$

Let x^v integer solution at node v

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \ \ \forall x$

Let x^v integer solution at node v

Let $\mathcal{I}_v$ the indices for which $x^v = 1$

Dealing with integers

Integer variables in the second stage (and binary first stage):

$$\phi \geq (Q(x^v) - L)|\sum_{i \in \mathcal{I}_v} x_i - \sum_{i \notin \mathcal{I}_v} x_i| - (Q(x^v) - L)(|\mathcal{I}_v| - 1) + L$$

Dealing with integers

Integer variables in the second stage (and binary first stage):

How does it work?

$$x = x^v \implies \phi \geq Q(x^v)$$

$$x \neq x^v \implies \phi \geq L^v \leq L$$

Dealing with integers

Integer variables in the second stage (and binary first stage):

The bound can be improved by looking in the neighborhood of x^v.

Classical (duality based) L-Shaped cuts on the LP relaxation help a lot!

Exercise

Exercise (approx. 50 min)

`https://tinyurl.com/sptutorial3`

Table of Contents

Feasibility

If $F^D(x^v, \xi_s) > 0$ for some s, let σ_s^v be its optimal solution. The feasibility cut

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

cuts off the second-stage-infeasible solution $x^v \notin \mathcal{K}_2$.

Proof.
Assume $x^v \notin \mathcal{K}_2 \rightarrow \exists\ s$ with $F^D(x^v, \xi_s) = F^P(x^v, \xi_s) > 0$

$$F^D(x^v, \xi_s) = (\sigma_s^v)^\top (h_s - T_s x^v) > 0$$

σ_s^v optimal to $F^D(x^v, \xi_s) \rightarrow x^v$ does not satisfy

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

□

Back

Feasibility

Solution $x^l \in \mathcal{K}_2$ satisfies feasibility cuts

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

Proof.
Assume $x^l \in \mathcal{K}_2$, then

$$F^D(x^l, \xi_s) = F^P(x^l, \xi_s) = 0 \qquad s = 1, \ldots, S$$

Solution σ_s^v to $F^D(x^v, \xi_s)$ is feasible for problem $F^D(x^l, \xi_s)$ but not optimal.

$$0 = F^D(x^l, \xi_s) = (\sigma_s^l)^\top (h_s - T_s x^l) \geq (\sigma_s^v)^\top (h_s - T_s x^l)$$

. Thus $x^l \in \mathcal{K}_2$ does not violate the feasibility cut. □

Back

Optimality

Proof optimality cuts.

Proof.
Assume $\phi^v < Q(x^v)$. Then we have

$$\phi^v < Q(x^v) = \sum_{s=1}^{S} \pi_s Q^D(x^v, \xi_s) = \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^v)$$

ρ_s^v optimal for $Q(x^v, \xi_s)$. Constraint

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

is not satisfied by (x^v, ϕ^v). □

Back

Optimality

Proof.
Assume $\phi^l \geq Q(x^l)$

$$\phi^l \geq Q(x^l) = \sum_{s=1}^{S} \pi_s Q^D(x^l, \xi_s) = \sum_{s=1}^{S} \pi_s (\rho_s^l)^\top (h_s - T_s x^l)$$

$$\sum_{s=1}^{S} \pi_s (\rho_s^l)^\top (h_s - T_s x^l) \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^l)$$

ρ_s^v is feasible for $Q^D(x^l, \xi_s)$ while ρ_s^l is optimal. Thus

$$\phi^l \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^l)$$

$\square$

Back