Stochastic Programming: A tutorial – Part I

DORS Tutorials 14/02/2023

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At the end of this tutorial you will know

How to formulate general stochastic programs

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What makes them difficult

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How to formulate general stochastic programs

What makes them difficult

How to solve them

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How to formulate general stochastic programs:two-stage linear (mixed-integer) recourse problems

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What makes them difficult and how to address the difficulty

How to solve them

At the end of this tutorial you will know

How to formulate general stochastic programs:two-stage linear (mixed-integer) recourse problems for a risk neutral decision maker

What makes them difficult and how to address the difficulty

How to solve (approximations of) two-stage stochastic programs

Limitations ...

- Risk-aversion
- Chance constraints
- Stochastic dominance
- Multi-stage problems
- Endogenous uncertainty
- Robust/Distributionally robust
- ...

Plan

- Introduction (45 min)
- break (15 min)
- Exercise 1 (10 min)
- More info on approximations (15 min)
- Exercise 2 (30 min)
- break (10 min)
- L-Shaped method (45 min)
- Exercise 3 (50 min)

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Uncertainty

Something is uncertain

$$(\Omega, \mathcal{F}, \mathbb{P})$$

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$$\omega \in \Omega \to \xi(\omega) \in \Xi$$

Uncertainty

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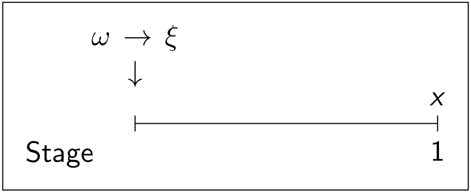
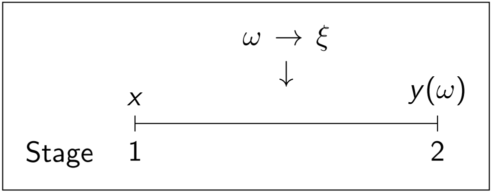
Notation can vary

Decision stages

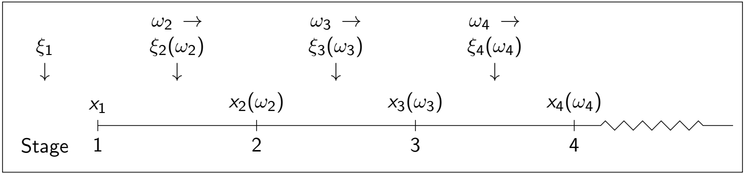


Decision stages

Decision stages



Decision stages



Two-Stage Stochastic Programs with Recourse

The decision maker:

1. Makes decisions $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$ (first decision stage)

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3. ω determines $\boldsymbol{\xi}(\omega)$ (our random data)

Two-Stage Stochastic Programs with Recourse

The decision maker:

1. Makes decisions $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$ (first decision stage)
2. Waits for the outcome $\omega \in \Omega$ of some random experiment.
3. ω determines $\boldsymbol{\xi}(\omega)$ (our random data)
4. Makes decisions $y(\omega) \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, given ξ and x (second decision stage)

Two-Stage Stochastic Programs with Recourse

$$\min \ c^\top x + \mathbb{E}_{\boldsymbol{\xi}}[\min \boldsymbol{q}(\omega)^T y(\omega)]$$

Two-Stage Stochastic Programs with Recourse

$$
\begin{aligned}
\min z = & c^\top x + \mathbb{E}_{\boldsymbol{\xi}}[\min \boldsymbol{q}(\omega)^T y(\omega)] \\
\text{s.t. } & Ax = b
\end{aligned}
$$

Two-Stage Stochastic Programs with Recourse

$$
\begin{aligned}
\min z = & c^\top x + \mathbb{E}_{\boldsymbol{\xi}}[\min \boldsymbol{q}(\omega)^T y(\omega)] \\
\text{s.t. } & Ax = b \\
& \boldsymbol{T}(\omega)x + \boldsymbol{W}(\omega)y(\omega) = \boldsymbol{h}(\omega) \quad a.s. \\
& x \in \mathcal{X}, y(\omega) \in \mathcal{Y}
\end{aligned}
$$

Two-Stage Stochastic Programs with Recourse

Parameters $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, and $A \in \mathbb{R}^{n_1 \times m_1}$ are known.

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$$\boldsymbol{\xi}(\omega) = \left(\boldsymbol{q}(\omega)^\top, \boldsymbol{h}(\omega)^\top, \boldsymbol{W}^{\mathbf{1}}(\omega), \ldots, \boldsymbol{W}(\omega)^{m_2}, \boldsymbol{T}(\omega)^1, \ldots, \boldsymbol{T}(\omega)^{m2}\right).$$

Two-Stage Stochastic Programs with Recourse

Parameters $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, and $A \in \mathbb{R}^{n_1 \times m_1}$ are known.

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ξ is a realization of $\boldsymbol{\xi}(\omega)$.

Two-Stage Stochastic Programs with Recourse

$$\begin{aligned} \min \;& c^\top x + Q(x) \\ \text{s.t. } & Ax = b \\ & x \in \mathcal{X} \end{aligned}$$

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$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}}[Q(x, \xi)]$$

Two-Stage Stochastic Programs with Recourse

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where

$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}}[Q(x, \xi)]$$

$$Q(x, \xi) = \min_y \{q^\top y | Wy = h - Tx, y \in \mathcal{Y}\}.$$

Two-Stage Stochastic Programs with Recourse with Discrete $\boldsymbol{\xi}$

Consider $\Xi = \{\xi_1, \ldots, \xi_S\}$ with probabilities π_s, $s = 1 \ldots, S$.

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$\xi_s \implies q_s, T_s, W_s, h_s$.

Two-Stage Stochastic Programs with Recourse with Discrete $\boldsymbol{\xi}$

Consider $\Xi = \{\xi_1, \dots, \xi_S\}$ with probabilities π_s, $s = 1 \dots, S$.

$\xi_s \implies q_s, T_s, W_s, h_s$.

$y(\omega)$ becomes $y_1, \dots, y_S$.

Two-Stage Stochastic Programs with Recourse with Discrete $\boldsymbol{\xi}$

$$
\begin{aligned}
\min\ & c^\top x + \sum_{s=1}^{S} \pi_s q_s^T y_s \\
\text{s.t. } & Ax = b \\
& T_s x + W_s y_s = h_s && s = 1, \ldots, S \\
& x \in \mathcal{X} \\
& y_s \in \mathcal{Y} && s = 1, \ldots, S.
\end{aligned}
$$

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A closer look at ξ

Where do we get ξ?

A closer look at ξ

Where do we get ξ?

- Number of failures $\approx$ Weibull
- Wind speed $\approx$ Weibull,Rayleigh
- Forecast error (linear regression) $\approx$ Normal
- Hospitalization in certain epidemics $\approx$ LogNormal
- Repair times $\approx$ LogNormal
- Choice model $\approx$ Gumbel, Normal, EV Type I
- Waiting times $\approx$ Beta
- ...

A closer look at ξ

Some randomness is discrete

A closer look at ξ

Some randomness is discrete

- Number of occurrences $\approx$ Poisson
- Number of trials before success $\approx$ Geometric
- Number of successes $\approx$ HyperGeometric
- ...

A closer look at ξ

High dimensions and co-dependencies are problematic

A closer look at the constraints

If ξ is continuous...

A closer look at the constraints

If ξ is continuous...

Constraints must hold a.s. ...

A closer look at the constraints

If ξ is continuous...

Constraints must hold a.s. ...

Possibly ∞ constraints

A closer look at $Q(x)$

The recourse function ...

$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}}\left[Q(x, \xi)\right] = \int_{\Omega} Q(x, \xi(\omega))\mathbb{P}(d\omega)$$

Why is this difficult?

A closer look at $Q(x)$

The recourse function ...

$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}}\big[Q(x,\xi)\big] = \int_{\Omega} Q(x,\xi(\omega))\mathbb{P}(d\omega)$$

Why is this difficult?

Ingredient 1: a closed form expression $Q(x,\xi)$

A closer look at $Q(x)$

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Ingredient 2: an antiderivative

A closer look at $Q(x)$

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Why is this difficult?

Ingredient 1: a closed form expression $Q(x,\xi)$

Ingredient 2: an antiderivative

Observe: $Q(x) = \int\int\cdots\int Q(x,\xi)\mathbb{D}(\xi)d\xi_1 d\xi_2\cdots d\xi_N$

A closer look at $Q(x)$

The recourse function ...

$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}}\big[Q(x, \xi)\big] = \int_{\Omega} Q(x, \xi(\omega))\mathbb{P}(d\omega)$$

Idea! Numerical integration!

A closer look at $Q(x)$

In one dimension (i.e., $N = 1$): Riemann sums, Trapezoidal rule, Simpson's rule

Ex. Riemann Sums

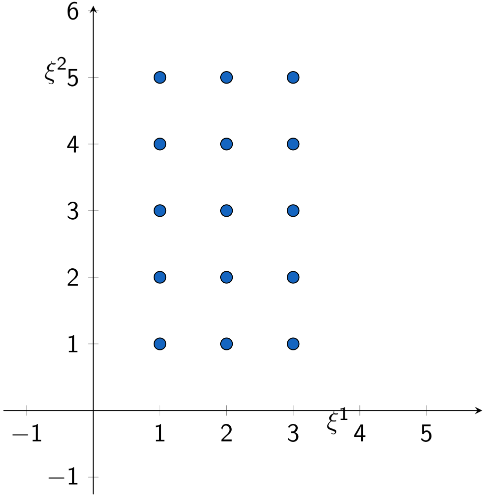
$$\int_a^b f(x)dx$$

- Partition $[a, b]$ using K points $x_0 = a$, $x_1, \ldots, x_K = b$ equally spaced Δx
- $\int_a^b f(x)dx \approx \sum_k f(x_k)\Delta x$
- As K increases we improve the approximation.

A closer look at $Q(x)$

In multiple dimensions: *Quadrature* methods.

Same principle, harder partition



A closer look at $Q(x)$

Numerical integration: it is already an approximation

A closer look at $Q(x)$

Numerical integration: it is already an approximation

all this work for one x...

A closer look at $Q(x)$

Numerical integration: it is already an approximation

all this work for one x...

we still have to solve the stochastic program..

A closer look at $Q(x)$

When $\boldsymbol{\xi}$ is discrete...

$$Q(x) = \sum_{s=1}^{S} \pi_s Q(x, \xi_s)$$

A closer look at $Q(x)$

When $\boldsymbol{\xi}$ is discrete...

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Finitely many linear constraints

A closer look at $Q(x)$

When $\boldsymbol{\xi}$ is discrete...

$$Q(x) = \sum_{s=1}^{S} \pi_s Q(x, \xi_s)$$

Finitely many linear constraints

When $\boldsymbol{\xi}$ is discrete ... but large ...

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Approximations

Continuous/discrete but large $\rightarrow$ discrete and small

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Continuous/discrete but large $\rightarrow$ discrete and small

Three categories of methods (loosely speaking)

- Probability Metrics
- Property Matching
- Monte Carlo

Method based on Probability Metrics in a nutshell

Results from research on stability, see,e.g.,
[Dup90, Pfl01, RR02, Röm03].

$$|z(\mathbb{P}) - z(\mathbb{Q})| \leq Ld(\mathbb{P}, \mathbb{Q})$$

Method based on Probability Metrics in a nutshell

- Start from large N scenarios (e.g., sampled)

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Method based on Probability Metrics in a nutshell

- Start from large N scenarios (e.g., sampled)
- Remove one scenario at a time to minimize the distance between the new and old distribution
- Add the probability of the deleted scenarios to the closest scenarios (in the sense of the probability metric)

Scenario reduction/generation, see, e.g., [HR03, DGKR03].

Property Matching in a nutshell

Idea: Replicate only the statistical properties that are important for the problem [HW01].

Create a small distribution that replicates only those properties.

Property Matching in a nutshell

Not always necessary to increase the size of the distribution.

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Not always necessary to increase the size of the distribution.

Problem driven.

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Observe: requires an NLP (heuristics exist [HKW03])

Property Matching in a nutshell

Not always necessary to increase the size of the distribution.

Problem driven.

Observe: requires an NLP (heuristics exist [HKW03])

Which properties?

Monte Carlo in a nutshell

Make K identical copies $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_K$ of $\boldsymbol{\xi}$.

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From each take, independently, a realization ξ_k.

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From each take, independently, a realization ξ_k.

Write the *Sample Average Approximation* (SAA)

$$
\begin{aligned}
z^K = \min f(x) :=& c^\top x + \sum_{k=1}^{K} \frac{1}{K} q_k^T y_k \\
\text{s.t. } & Ax = b, \\
& T_k x + W_k y_k = h_k, \qquad k = 1, \ldots, K \\
& x \in \mathcal{X}, \\
& y_k \in \mathcal{Y}, \qquad k = 1, \ldots, K.
\end{aligned}
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Sample Average Approximation (SAA)

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& T_k x + W_k y_k = h_k, \qquad k = 1, \dots, K \\
& x \in \mathcal{X}, \\
& y_k \in \mathcal{Y}, \qquad k = 1, \dots, K.
\end{aligned}
$$

or

$$
\begin{aligned}
z^K = \min f^K(x) :=& c^\top x + \sum_{k=1}^{K} \frac{1}{K} Q(x, \xi_k) \\
\text{s.t. } & Ax = b, \\
& x \in \mathcal{X}
\end{aligned}
$$

SAA: some observations

z^K is a stochastic estimator of z^*

SAA: some observations

$$f^K(\bar{x}) = c^\top \bar{x} + \sum_{k=1}^{K} \frac{1}{K} Q(\bar{x}, \xi_k)$$

is an **unbiased** estimator (pointwise $x = \bar{x}$) of

$$f(\bar{x}) = c^\top \bar{x} + \mathbb{E}_{\boldsymbol{\xi}}[Q(\bar{x}, \xi)]$$

SAA: some observations

Finally, $f^K(\bar{x})$ a *consistent estimator* of $f(\bar{x})$, that is

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} Q(\bar{x},\xi_k) \to \mathbb{E}_{\boldsymbol{\xi}}\left[Q(\bar{x},\xi)\right] = Q(\bar{x})$$

We say that $f^K(\bar{x})$ *converges pointwise*.

SAA: some observations

Finally, $f^K(\bar{x})$ a *consistent estimator* of $f(\bar{x})$, that is

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} Q(\bar{x},\xi_k) \to \mathbb{E}_{\boldsymbol{\xi}}\left[Q(\bar{x},\xi)\right] = Q(\bar{x})$$

We say that $f^K(\bar{x})$ *converges pointwise*.

But observe

$$\mathbb{S}td[\frac{1}{K}\sum_{k=1}^{K} Q(\bar{x},\xi_k)] = \frac{\mathbb{S}td[Q(\bar{x},\xi)]}{\sqrt{K}}$$

SAA: some observations

Exercise 1: 15 minutes

`https://tinyurl.com/sptutorial1`

SAA: some observations

Under certain conditions $z^K \to z^*$ as $k \to \infty$ (exponentially fast!)

SAA: some observations

Under certain conditions $z^K \to z^*$ as $k \to \infty$ (exponentially fast!)

$\mathbb{E}[z^K]$ gives a statistical lower bound! (obs! z^K is biased)

Proof

SAA: some observations

Under certain conditions $z^K \to z^*$ as $k \to \infty$ (exponentially fast!)

$\mathbb{E}[z^K]$ gives a statistical lower bound! (obs! z^K is biased)

Proof

$\mathbb{E}\left[c^\top \bar{x} + \frac{1}{K}\sum_{k=1}^{K}[Q(\bar{x}, \xi_k)]\right]$ gives a statistical upper bound!

See, e.g., [Sha91, MMW99, Sha03].

SAA: some observations

Exercise 2: 30 minutes

`https://tinyurl.com/sptutorial2`

Approximations

Getting a good solution vs Estimating its value

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Mathematical properties of discrete stochastic programs

$$\mathcal{K}_2(\xi) = \{x | \exists y \geq 0, \text{s.t.} W(\omega)y = h(\omega) - T(\omega)x\}$$

Mathematical properties of discrete stochastic programs

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Convex and polyhedral!

Mathematical properties of discrete stochastic programs

$$\mathcal{K}_2 = \bigcap_{\xi \in \Xi} \mathcal{K}_2(\xi)$$

Mathematical properties of discrete stochastic programs

$$\mathcal{K}_2 = \bigcap_{\xi \in \Xi} \mathcal{K}_2(\xi)$$

Convex and polyhedral!

Mathematical properties of discrete stochastic programs

Useful jargon

Complete recourse

$$\mathcal{K}_2 = \mathbb{R}^{n_1}$$

Mathematical properties of discrete stochastic programs

Useful jargon

Complete recourse

$$\mathcal{K}_2 = \mathbb{R}^{n_1}$$

Relatively complete recourse

$$\mathcal{K}_2 \subseteq \mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$$

.

Mathematical properties of discrete stochastic programs

$Q(x, \xi)$ is:

a. piece-wise linear convex in h, T and x,

b. piece-wise linear concave in q.

Mathematical properties of discrete stochastic programs

$$Q(x) = \mathbb{E}_{\boldsymbol{\xi}} Q(x, \xi) = \sum_{s=1}^{S} \pi_s Q(x, \xi_s)$$

piece-wise linear convex in x.

Proof of lower bound

The expectation of the optimal objective value of SAAs of size K is a lower bound for the true optimal objective value, that is:

$$\mathbb{E}[z^K] \leq z^*$$

Proof.
First, observe that $z^* = \min_{x \in \mathcal{K}_1} f(x) = \min_{x \in \mathcal{K}_1} c^\top x + \mathbb{E}\left[K^{-1} \sum_{k=1}^{K} Q(x, \xi_k)\right]$. We can use a direct proof. We know that for all $x' \in \mathcal{K}_1$

$$\min_{x \in \mathcal{K}_1} K^{-1} \sum_{k=1}^{K} c^\top x + Q(x, \xi_k) \leq K^{-1} \sum_{k=1}^{K} c^\top x' + Q(x', \xi_k)$$

by taking the expectation of both sides we obtain continues... □

Proof of lower bound

The expectation of the optimal objective value of SAAs of size K is a lower bound for the true optimal objective value, that is:

$$\mathbb{E}[z^K] \leq z^*$$

Proof.

continuing...

$$\mathbb{E}\left[\min_{x \in \mathcal{K}_1} K^{-1} \sum_{k=1}^{K} c^\top x + Q(x, \xi_k)\right] \leq \mathbb{E}\left[K^{-1} \sum_{k=1}^{K} c^\top x' + Q(x', \xi_k)\right]$$

which corresponds to

$$\mathbb{E}\left[z^K\right] \leq \mathbb{E}\left[K^{-1} \sum_{k=1}^{K} c^\top x' + Q(x', \xi_k)\right]$$

continues... □

Proof of lower bound

The expectation of the optimal objective value of SAAs of size K is a lower bound for the true optimal objective value, that is:

$$\mathbb{E}[z^K] \leq z^*$$

Proof.
continuing... Now, since the inequality holds for all x', it holds also for the solution x' that minimizes expectation on the right-hand-side and which yields z^*, that is

$$\mathbb{E}\left[z^K\right] \leq \min_{x' \in \mathcal{K}_1} \mathbb{E}\left[K^{-1} \sum_{k=1}^{K} c^\top x' + Q(x', \xi_k)\right] = z^*$$

□

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