# L-Shaped method

Exercise

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# Consider the following problem

$$\begin{aligned} \min\ & 2x + \mathbb{E}\big[y_1 + 3y_2\big] \\ \text{s.t. } & 2y_1 + y_2 + x = \boldsymbol{h}_1 \quad a.s. \\ & -y_1 + y_2 + 4x \geq \boldsymbol{h}_2 \quad a.s. \\ & y_1, y_2, x \geq 0 \end{aligned}$$

$(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \Xi = \big\{(1,2),(1,3),(2,1)\big\}$ Equal probability.

# Consider the following problem

$$\begin{aligned}
\min\ & 2x + \sum_{s=1}^{3} \pi_s \big[ y_{1s} + 3y_{2s} \big] & \\
\text{s.t. } & 2y_{1s} + y_{2s} + x = h_{1s} & s = 1, \ldots, 3 \\
& -y_{1s} + y_{2s} + 4x \geq h_{2s} & s = 1, \ldots, 3 \\
& y_{1s}, y_{2s} \geq 0 & s = 1, \ldots, 3 \\
& x \geq 0 &
\end{aligned}$$

$$(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \Xi = \big\{ (1,2), (1,3), (2,1) \big\}$$

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

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# Master Problem

Iteration $v = 1$

$$MP^v = \min\ 2x + \phi$$
$$\text{s.t. } x, \phi \geq 0$$

Solution $(x^v, \phi^v)$

# Feasibility Subproblem

$$
\begin{aligned}
F(x^v, \xi_s) = \min \;\; & v_1 + v_2 + v_3 \\
\text{s.t. } & 2y_1 + y_2 + v_1 - v_2 = h_{1s} - x^v \\
& -y_1 + y_2 + v_3 \geq h_{2s} - 4x^v \\
& y_1, y_2 \geq 0
\end{aligned}
$$

# Optimality Subproblem

$$
\begin{aligned}
Q(x^v, \xi_s) = \min\ & y_1 + 3y_2 \\
\text{s.t. } & 2y_1 + y_2 = h_{1s} - x^v \\
& -y_1 + y_2 \geq h_{2s} - 4x^v \\
& y_1, y_2 \geq 0
\end{aligned}
$$

# Iteration 1

- Solution $MP^1$ $(x^1, \phi^1) = (0, 0)$
- $F(x^1, \xi_1) = 1$, $(\sigma_1, \sigma_2) = (-1, 1)$
- $F(x^1, \xi_2) = 2$, $(\sigma_1, \sigma_2) = (-1, 1)$
- $F(x^1, \xi_3) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$

Feasibility cuts

$$-1(1 - x) + 1(2 - 4x) \leq 0$$

$$-1(1 - x) + 1(3 - 4x) \leq 0$$

# Iteration 2

- Solution $MP^2$ $(x^2, \phi^2) = (0.66666, 0)$
- $F(x^2, \xi_1) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^2, \xi_2) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^2, \xi_3) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$

$x^2 \in \mathcal{K}_2$ (second-stage feasible)

# Iteration 2

- Solution $MP^2$ $(x^2, \phi^2) = (0.66666, 0)$
- $Q(x^2, \xi_1) = 0.16$, $(\rho_1, \rho_2) = (0.5, 0)$
- $Q(x^2, \xi_2) = 1.0$, $(\rho_1, \rho_2) = (1.33, 1.66)$
- $Q(x^2, \xi_2) = 0.66$, $(\rho_1, \rho_2) = (0.5, 0)$

Since $0 = \phi^2 < Q(x^2) = 0.66666$ we add the following optimality cut

$$\begin{aligned}\phi \geq & \frac{1}{3}0.5(1-x) + 0(2-4x) + \\ & \frac{1}{3}1.33(1-x) + 1.66(3-4x) + \frac{1}{3}0.5(2-x) + 0(1-4x)\end{aligned}$$

# Iteration 3

- Solution $MP^3$ $(x^3, \phi^3) = (0.87037, 0)$
- $F(x^3, \xi_1) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^3, \xi_2) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^3, \xi_3) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$

$x^3 \in \mathcal{K}_2$ (second-stage feasible)

# Iteration 3

- Solution $MP^3$ $(x^3, \phi^3) = (0.87037, 0)$
- $Q(x^3, \xi_1) = 0.06481$, $(\rho_1, \rho_2) = (0.5, 0)$
- $Q(x^3, \xi_2) = 0.06481$, $(\rho_1, \rho_2) = (0.5, 0)$
- $Q(x^3, \xi_2) = 0.56481$, $(\rho_1, \rho_2) = (0.5, 0)$

Since $0 = \phi^3 < Q(x^3) = 0.23148$ we add the following optimality cut

$$\begin{aligned}\phi \geq & \frac{1}{3}0.5(1-x) + 0(2-4x) + \\ & \frac{1}{3}0.5(1-x) + 0(3-4x) + \frac{1}{3}0.5(2-x) + 0(1-4x)\end{aligned}$$

# Iteration 4

- Solution $MP^4$ $(x^4, \phi^4) = (0.7777, 0.27777)$
- $F(x^4, \xi_1) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^4, \xi_2) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$
- $F(x^4, \xi_3) = 0$, $(\sigma_1, \sigma_2) = (0, 0)$

$x^3 \in \mathcal{K}_2$ (second-stage feasible)

# Iteration 4

- Solution $MP^4$ $(x^4, \phi^4) = (0.7777, 0.27777)$
- $Q(x^4, \xi_1) = 0.11111$, $(\rho_1, \rho_2) = (0.5, 0)$
- $Q(x^4, \xi_2) = 0.11111$, $(\rho_1, \rho_2) = (0.5, 0)$
- $Q(x^4, \xi_2) = 0.61111$, $(\rho_1, \rho_2) = (0.5, 0)$

Since $0.27777 = \phi^4 = Q(x^4) = 0.27777$, solution $x^4 = 0.7777$ is optimal, with optimal objective value $MP^4 = 1.8333$.