Stochastic Programming

Solution Methods

Giovanni Pantuso

Department of Mathematical Sciences
University of Copenhagen
Copenhagen, Denmark

Table of Contents

Applicability

Two-stage linear stochastic programs with recourse where

- ξ is a discrete random variable,
- $\mathcal{X} = \mathbb{R}_+^{n_1}$,
- $\mathcal{Y} = \mathbb{R}_+^{n_2}$.

The integer case requires some adjustments.

Recall

The deterministic equivalent problem

$$\begin{aligned} \min z = c^T x + Q(x) \\ \text{s.t.} Ax = b \\ x \geq 0 \end{aligned}$$

where

$$Q(x) = \sum_{s=1}^{S} \pi_s Q(x, \xi_s)$$

and

$$Q(x, \xi_s) = \min_y \{q_s^T y | W_s y = h_s - T_s x, y \geq 0\}.$$

Recall

We call $\mathcal{K}_1 = \{x | Ax = b, x \geq 0\}$ When $\mathcal{Y} = \mathbb{R}_+^{n_2}$ and $\boldsymbol{\xi}$ is discrete:

- $Q(x)$ is piecewise linear and convex in x
- $\mathcal{K}_2$ is a closed and convex polyhedron

This will help..

A reformulation of the DEP

$$\min z = c^T x + Q(x)$$
$$\text{s.t.}\, x \in \mathcal{K}_1 \cap \mathcal{K}_2$$

A reformulation of the DEP

If we introduce a variable ϕ we can obtain another reformulation

$$\begin{aligned} \min z &= c^T x + \phi \\ \text{s.t.} x &\in \mathcal{K}_1 \\ x &\in \mathcal{K}_2 \\ \phi &\geq Q(x) \end{aligned}$$

A reformulation of the DEP

A reformulation of the DEP

Polyhedral formulation, but with way too many constraints..

Idea! Drop $x \in \mathcal{K}_2$ and $\phi \geq Q(x)$ and reconstruct them iteratively... (We may not need all of their constraints).

The Master Problem

At a generic iteration..

$$\begin{aligned}
\min z &= c^T x + \phi \\
\text{s.t.} x &\in \mathcal{K}_1 \\
f_i(x) &\le 0 && i = 1, \ldots, I, \\
g_j(x, \phi) &\le 0 && j = 1, \ldots, J
\end{aligned}$$

The Master Problem

At a generic iteration..

$$\begin{aligned} \min z = c^T x + \phi & \\ \text{s.t.} x \in \mathcal{K}_1 & \\ f_i(x) \le 0 & \qquad i = 1, \dots, I, \\ g_j(x, \phi) \le 0 & \qquad j = 1, \dots, J \end{aligned}$$

Initially $I = J = 0$.

Feasibility

At iteration v we solve MP and find (x^v, ϕ^v).

Does $x^v \in \mathcal{K}_2$*?* Let's check:

For each s we solve the *feasibility subproblem*.

Feasibility

$$F^P(x^v, \xi_s) = \min_{y, v^+, v^-} e^\top v^+ + e^\top v^-$$

$$\text{s.t.}\, W_s y + I v^+ - I v^- = h_s - T_s x^v,$$

$$y, v^+, v^- \geq 0$$

where $e^\top = (1, \ldots, 1)$ and I is the identity matrix.

Find the differences:

$$Q(x, \xi_s) = \min_y \{ q_s^T y | W_s y = h_s - T_s x, y \geq 0 \}.$$

Feasibility

$$F^P(x^v, \xi_s) = \min_{y, v^+, v^-} \{e^\top v^+ + e^\top v^- | W_s y + I v^+ - I v^- = h_s - T_s x^v, y, v^+, v^- \geq 0\}$$

Its dual

$$F^D(x^v, \xi_s) = \max_{\sigma} \{\sigma^\top (h_s - T_s x^v) | \sigma^\top W_s \leq 0, \sigma^\top I \leq e^\top, -\sigma^\top I \leq e^\top\}$$

Both are always feasible. Strong duality $F^D(x^v, \xi_s) = F^P(x^v, \xi_s)$.

Feasibility

If $F^P(x^v, \xi_s) = F^D(x^v, \xi_s) = 0$ for all s then $x^v \in \mathcal{K}_2$ otherwise it does not.

If $x^v \notin \mathcal{K}_2$ we need to tell MP that x^v is not a good solution and must be cut off.

Feasibility

If $F^D(x^v, \xi_s) > 0$ for some s, let σ_s^v be its optimal solution. The feasibility cut

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

cuts off the second-stage-infeasible solution $x^v \notin \mathcal{K}_2$.

Proof

Feasibility

Adding

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

to MP will cut off solution x^v at the next iteration.

Feasibility

Solution $x^l \in \mathcal{K}_2$ satisfies feasibility cuts

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

Proof

Feasibility

Summary:

- we know how verify $x^v \in \mathcal{K}_2$,
- we know that $(\sigma_s^v)^\top (h_s - T_s x) \leq 0$ will cut off infeasible solutions,
- we know that $(\sigma_s^v)^\top (h_s - T_s x) \leq 0$ will not cut off feasible solutions.

Optimality

Assume (x^v, ϕ^v) is now such that

$$x^v \in \mathcal{K}_2$$

. We should now verify whether

$$\phi^v \geq Q(x^v)$$

. We need to calculate

$$Q(x^v) = \sum_{s=1}^{S} \pi_s Q(x^v, \xi_s)$$

Optimality

For $s = 1, \ldots, S$ solve

$$Q^P(x^v, \xi_s) = \min_y \{q_s^\top y | W_s y = h_s - T_s x^v, y \geq 0\}$$

or its dual

$$Q^D(x^v, \xi_s) = \max_\rho \{\rho^\top (h_s - T_s x^v) | \rho^\top W_s \leq q_s^\top\}$$

Optimality

Observe:

- $Q^P(x^v, \xi_s)$ is feasible (and, we assume, bounded)
- $Q^P(x^v, \xi_s) = Q^D(x^v, \xi_s)$,
- $Q(x^v) = \sum_{s=1}^{S} \pi_s Q^P(x^v, \xi_s) = \sum_{s=1}^{S} \pi_s Q^D(x^v, \xi_s)$.

Optimality

If $\phi^v < Q(x^v)$, then (x^v, ϕ^v) is cut off by optimality cut

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

where ρ_s^v is the optimal solution to $Q^D(x^v, \xi_s)$. Proof

Optimality

(x^l, ϕ^l), such that $\phi^l \geq Q(x^l)$, satisfies

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

Proof

Optimality

Summarizing:

- We know how to check optimality,
- We know how to cut off (x^v, ϕ^v) such that $\phi^v < Q(x^v)$,
- We know that optimality cuts preserve (x^l, ϕ^l) such that $\phi^l \geq Q(x^l)$.

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.
4. For $s = 1, \ldots, S$ solve $Q^D(x^v, \xi_s)$ and calculate $Q(x^v)$

Putting everything together

1. Solve MP (initially no cuts) to find (x^v, ϕ^v)
2. For $s = 1, \ldots, S$ solve $F^D(x^v, \xi_s)$
3. If $F^D(x^v, \xi_s) > 0$ for some s, add a feasibility cut and return to STEP 1.
4. For $s = 1, \ldots, S$ solve $Q^D(x^v, \xi_s)$ and calculate $Q(x^v)$
5. If $\phi^v \geq Q(x^v)$, STOP (x^v, ϕ^v) is optimal otherwise add an optimality cut and return to STEP 1.

A finite algorithm

The algorithm converges

- finitely many possible cuts
- if (at most) all cuts are available, the solution to MP is optimal.

Bounds

$$c^\top x^v + \phi^v \leq z^* \leq c^\top x^v + Q(x^v)$$

Dealing with integers

Integer variables in the first stage

VS

Integer variables in the second stage

Dealing with integers

Integer variables in the first stage:

Embed the L-Shaped Method into Branch and Bound.

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \forall x$

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \forall x$

Let x^v integer solution at node v

Dealing with integers

Integer variables in the second stage (and binary first stage):

Let $L \leq Q(x) \forall x$

Let x^v integer solution at node v

Let $\mathcal{I}_v$ indices for which $x^v = 1$

Dealing with integers

Integer variables in the second stage (and binary first stage):

$$\phi \geq (Q(x^v) - L)|\sum_{i \in \mathcal{I}_v} x_i - \sum_{i \notin \mathcal{I}_v} x_i| - (Q(x^v) - L)(|\mathcal{I}_v| - 1) + L$$

Dealing with integers

Integer variables in the second stage (and binary first stage):

How does it work?

$$x = x^v \implies \phi \geq Q(x^v)$$

$$x \neq x^v \implies \phi \geq L^v \leq L$$

Dealing with integers

Integer variables in the second stage (and binary first stage):

The bound can be improved by looking in the neighborhood of x^v.

Classical (duality based) L-Shaped cuts on the LP relaxation help a lot!

Table of Contents

Applicability

Multistage stochastic programs (possibly integer at all stages)

Applicability

Multistage stochastic programs (possibly integer at all stages)

- $\boldsymbol{\xi}$ is a discrete random variable (assume not too large)
- $\mathcal{X}_t$ may contain integrality restrictions on all/some decision variables.

In a nutshell

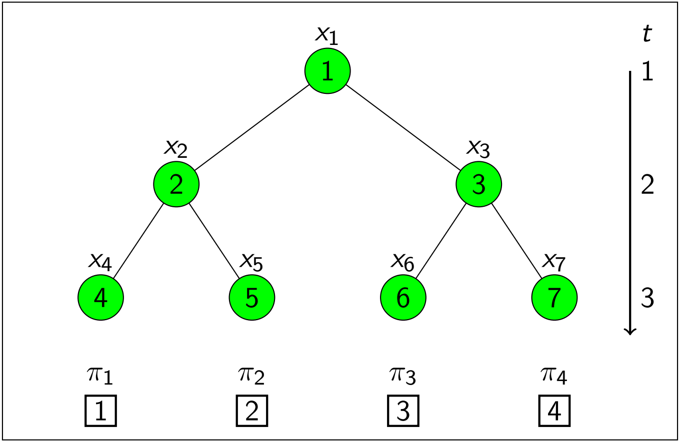
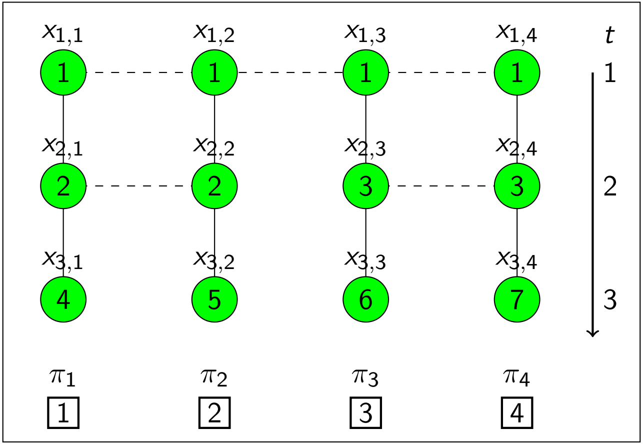


Figure 1

In a nutshell



In a nutshell

- Use a scenario formulation

In a nutshell

- Use a scenario formulation
- Relax NACs (Lagrangian Relaxation)

In a nutshell

- Use a scenario formulation
- Relax NACs (Lagrangian Relaxation)
- Use the Lagrangian bound in a Branch and Bound framework

In a nutshell

- Use a scenario formulation
- Relax NACs (Lagrangian Relaxation)
- Use the Lagrangian bound in a Branch and Bound framework
- Branch until NACs are reconstructed

Reformulation

Assume a two-stage SP

$$\mathcal{S}_s = \{(x, y_s) : x \in \mathcal{K}_1, x \in \mathcal{X},\, T_s x + W_s y_s = h_s, y_s \in \mathcal{Y}\}$$

We can write the two-stage stochastic program as follows

$$z^* = \min \left\{ c^\top x + \sum_{s=1}^{S} \pi_s q_s^\top y_s : (x, y_s) \in \mathcal{S}_s, s = 1, \ldots, S \right\}$$

Reformulation

$$z^* = \min \left\{ c^\top x + \sum_{s=1}^{S} \pi_s q_s^\top y_s : (x, y_s) \in \mathcal{S}_s, s = 1, \ldots, S \right\}$$

$$z^* = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) \middle| \begin{array}{ll} (x_s, y_s) \in \mathcal{S}_s, & s = 1, \ldots, S \\ x_1 = x_2 = \cdots & = x_{S-1} = x_S \end{array} \right\}$$

Reformulation

$$z^* = \min \left\{ c^\top x + \sum_{s=1}^{S} \pi_s q_s^\top y_s : (x, y_s) \in \mathcal{S}_s, s = 1, \dots, S \right\}$$

$$z^* = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) \middle| \begin{array}{ll} (x_s, y_s) \in \mathcal{S}_s, & s = 1, \dots, S \\ x_1 = x_2 = \cdots & = x_{S-1} = x_S \end{array} \right\}$$

$$x_1 = x_2 = \cdots = x_{S-1} = x_S \implies \sum_{s=1}^{S} H_s x_s = 0$$

Lagrangian Relaxation

$$z^* = \min \left\{ \sum_{s=1}^{S} \pi_s \big(c^\top x_s + q_s^\top y_s \big) \;\middle|\; \begin{array}{ll} (x_s, y_s) \in \mathcal{S}_s, & s = 1, \ldots, S \\ \sum_{s=1}^{S} H_s x_s = 0 & \end{array} \right\}$$

Lagrangian Relaxation

$$z^* = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) \middle| \begin{array}{cc} (x_s, y_s) \in \mathcal{S}_s, & s = 1, \dots, S \\ \sum_{s=1}^{S} H_s x_s = 0 & \end{array} \right\}$$

For given λ

$$D(\lambda) = \min_{x,y} \left\{ \sum_{s=1}^{s} \left[\pi_s \left(c^\top x_s + q_s^\top y_s \right) + \lambda H_s x_s \right] : (x_s, y_s) \in \mathcal{S}_s, s = 1, \dots, S \right\}$$

Lagrangian Relaxation

For all λ,

$$D(\lambda) \leq z^*$$

Proof

The Lagrangian Dual

$$z_{LD} = \max_{\lambda} D(\lambda)$$

The Lagrangian Dual

$$z_{LD} = \max_{\lambda} D(\lambda)$$

$$z_{LD} \leq z^*$$

We can close the gap!

If for some choice of λ the solution $(x_s, y_s)_{s=1}^{S}$ to $D(\lambda)$ is feasible for the stochastic program, then

We can close the gap!

If for some choice of λ the solution $(x_s, y_s)_{s=1}^S$ to $D(\lambda)$ is feasible for the stochastic program, then

- $(x_s, y_s)_{s=1}^S$ is an optimal solution to the stochastic program,

We can close the gap!

If for some choice of λ the solution $(x_s, y_s)_{s=1}^S$ to $D(\lambda)$ is feasible for the stochastic program, then

- $(x_s, y_s)_{s=1}^S$ is an optimal solution to the stochastic program,
- λ is an optimal solution to the Lagrangian dual.

Proof

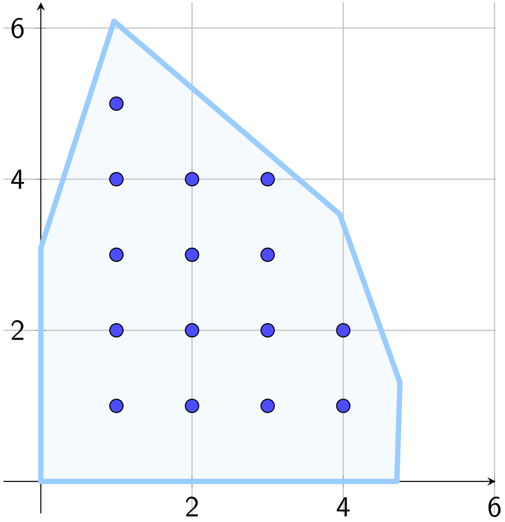
Usually, we are not so lucky

However,

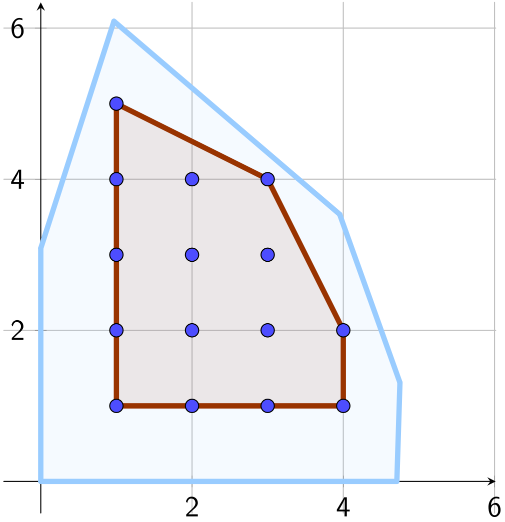
$$z_{LD} = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) \middle| \begin{array}{l} (x_s, y_s) \in \text{conv}\mathcal{S}_s, \quad s = 1, \dots, S \\ x_1 = \cdots = x_S \end{array} \right\}$$

Proof

But usually we do not close the gap...



But usually we do not close the gap..



So what?

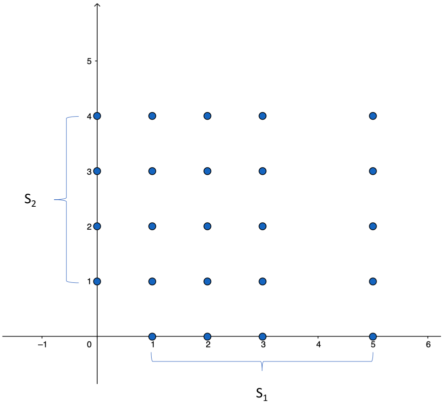
The feasible region of

$$z_{LD} = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) : (x_s, y_s) \in conv \mathcal{S}_s, s = 1, \ldots, S, x_1 = \cdots = x_S \right\}$$

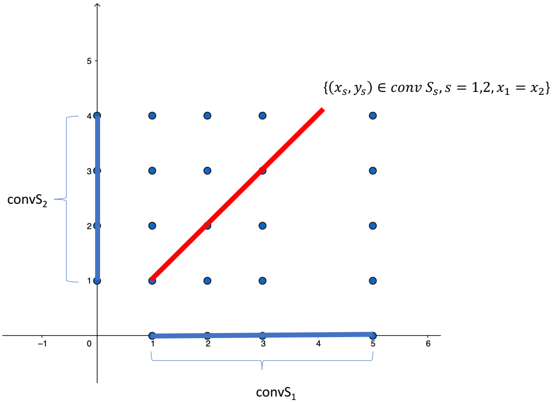
Contains

$$z^* = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) : conv \left\{ (x_s, y_s) \in \mathcal{S}_s, s = 1, \ldots, S, x_1 = \cdots = x_S \right\} \right\}$$

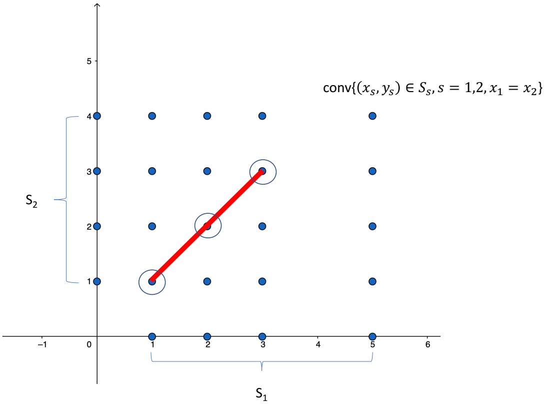
So what?



So what?



So what?



So what?

Nevertheless, the feasible region of

$$z_{LD} = \min\left\{\sum_{s=1}^{S}\pi_s\big(c^\top x_s + q_s^\top y_s\big) : (x_s, y_s) \in conv\mathcal{S}_s, s = 1, \ldots, S, x_1 = \cdots = x_S\right\}$$

However it is contained in the feasible region of

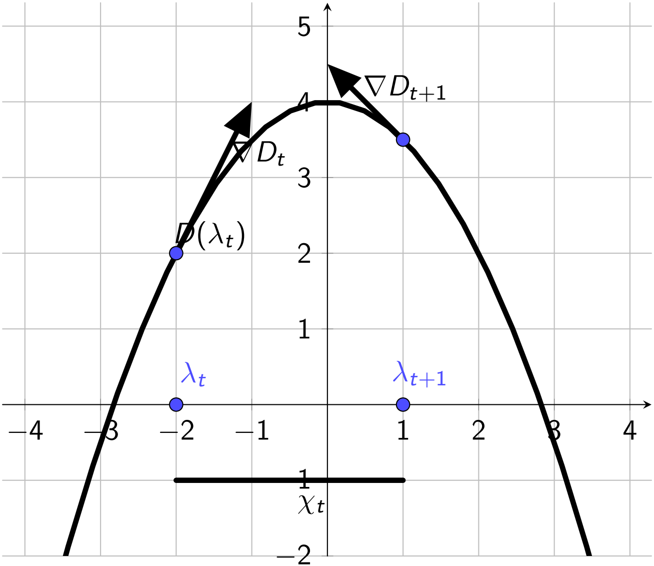
$$z_{LP} = \min\left\{\sum_{s=1}^{S}\pi_s c^\top x_s + q_s^\top y_s : (x_s, y_s) \in \mathcal{S}_s^{LP}, s = 1, \ldots, S, x_1 = \cdots = x_S\right\}$$

How do we solve the dual?

$D(\lambda)$ is concave in λ.

Proof

How do we solve the dual?



How do we solve the dual?

$D(\lambda)$ splits into S independent problems

$$D(\lambda) = \min_{x,y} \left\{ \sum_{s=1}^{S} \left[\pi_s \left(c^\top x_s + q_s^\top y_s \right) + \lambda H_s x_s \right] : (x_s, y_s) \in \mathcal{S}_s, s = 1, \ldots, S \right\}$$

Thus, at every iteration of the sub-gradient method we solve S smaller problems.

A Branch and Bound algorithm

So far it is clear that:

- In general we observe a duality gap ($z_{LD} < z^*$)
- The duality gap emerges because NACs are violated
- $z_{LD} \geq z_{LP}$

Idea: use Branch and Bound to fix NACs!

A Branch and Bound algorithm

STEP 1 Set $\bar{z} = +\infty$ and $\mathcal{P}$ contains only the original stochastic program.

A Branch and Bound algorithm

STEP 1 Set $\bar{z} = +\infty$ and $\mathcal{P}$ contains only the original stochastic program.

STEP 2 If $\mathcal{P} = \emptyset$ STOP, solution $(\bar{x}, \bar{y})$, which yielded $\bar{z} = c^\top \bar{x} + Q(\bar{x})$ is optimal.

A Branch and Bound algorithm

STEP 1 Set $\bar{z} = +\infty$ and $\mathcal{P}$ contains only the original stochastic program.

STEP 2 If $\mathcal{P} = \emptyset$ STOP, solution $(\bar{x}, \bar{y})$, which yielded $\bar{z} = c^\top \bar{x} + Q(\bar{x})$ is optimal.

STEP 3 Select and delete a node P from $\mathcal{P}$ and solve its Lagrangian dual whose optimal objective yields $z_{LD}(P)$. If P is infeasible go to STEP 2.

A Branch and Bound algorithm

STEP 4 If $z_{LD}(P) \geq \bar{z}$ go to STEP 2. Otherwise, let $(x_s^P, y_s^P)_{s=1}^S$ be the solution to the dual.

A Branch and Bound algorithm

STEP 4 If $z_{LD}(P) \geq \bar{z}$ go to STEP 2. Otherwise, let $(x_s^P, y_s^P)_{s=1}^S$ be the solution to the dual.

4.A If $x_1^P = \cdots = x_S^P$

Update $\bar{z}$ if possible

Delete from $\mathcal{P}$ all problems P' with $z_{LD}(P') \geq \bar{z}$

Go to STEP 2.

A Branch and Bound algorithm

STEP 4 If $z_{LD}(P) \geq \bar{z}$ go to STEP 2. Otherwise, let $(x_s^P, y_s^P)_{s=1}^S$ be the solution to the dual.

A Branch and Bound algorithm

STEP 4 If $z_{LD}(P) \geq \bar{z}$ go to STEP 2. Otherwise, let $(x_s^P, y_s^P)_{s=1}^S$ be the solution to the dual.

4.B If the x_s^P solutions are different:
Compute their average $\hat{x}^P = \sum_{s=1}^S \pi_s x_s^P$

A Branch and Bound algorithm

STEP 5 Select a component x^i of x and add two new problems to $\mathcal{P}$, that is $P \cup \{x_s^i \leq \lfloor \hat{x}_i^P \rfloor\}$ and $P \cup \{x_s^i \geq \lfloor \hat{x}_i^P \rfloor + 1\}$.

Table of Contents

Feasibility

If $F^D(x^v, \xi_s) > 0$ for some s, let σ_s^v be its optimal solution. The feasibility cut

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

cuts off the second-stage-infeasible solution $x^v \notin \mathcal{K}_2$.

Proof.
Assume $x^v \notin \mathcal{K}_2 \rightarrow \exists\ s$ with $F^D(x^v, \xi_s) = F^P(x^v, \xi_s) > 0$

$$F^D(x^v, \xi_s) = (\sigma_s^v)^\top (h_s - T_s x^v) > 0$$

σ_s^v optimal to $F^D(x^v, \xi_s) \rightarrow x^v$ does not satisfy

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

□

Back

Feasibility

Solution $x^l \in \mathcal{K}_2$ satisfies feasibility cuts

$$(\sigma_s^v)^\top (h_s - T_s x) \leq 0$$

Proof.
Assume $x^l \in \mathcal{K}_2$, then

$$F^D(x^l, \xi_s) = F^P(x^l, \xi_s) = 0 \qquad s = 1, \dots, S$$

Solution σ_s^v to $F^D(x^v, \xi_s)$ is feasible for problem $F^D(x^l, \xi_s)$ but not optimal.

$$0 = F^D(x^l, \xi_s) = (\sigma_s^l)^\top (h_s - T_s x^l) \geq (\sigma_s^v)^\top (h_s - T_s x^l)$$

. Thus $x^l \in \mathcal{K}_2$ does not violate the feasibility cut. □

Back

Optimality

Proof optimality cuts.

Proof.
Assume $\phi^v < Q(x^v)$. Then we have

$$\phi^v < Q(x^v) = \sum_{s=1}^{S} \pi_s Q^D(x^v, \xi_s) = \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^v)$$

ρ_s^v optimal for $Q(x^v, \xi_s)$. Constraint

$$\phi \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x)$$

is not satisfied by (x^v, ϕ^v). □

Back

Optimality

Proof.

Assume $\phi^l \geq Q(x^l)$

$$\phi^l \geq Q(x^l) = \sum_{s=1}^{S} \pi_s Q^D(x^l, \xi_s) = \sum_{s=1}^{S} \pi_s (\rho_s^l)^\top (h_s - T_s x^l)$$

$$\sum_{s=1}^{S} \pi_s (\rho_s^l)^\top (h_s - T_s x^l) \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^l)$$

ρ_s^v is feasible for $Q^D(x^l, \xi_s)$ while ρ_s^l is optimal. Thus

$$\phi^l \geq \sum_{s=1}^{S} \pi_s (\rho_s^v)^\top (h_s - T_s x^l)$$

□

Lagrangian Relaxation

For all λ, $D(\lambda) \leq z^*$

Proof.
Take $(x_s^*, y_s^*)_{s=1}^S$ and an arbitrary $\hat{\lambda}$. We can write

$$z^* = \sum_{s=1}^S \pi_s \big(c^\top x_s^* + q_s^\top y_s^*\big) = \sum_{s=1}^S \pi_s \big(c^\top x_s^* + q_s^\top y_s^*\big) + \hat{\lambda} \underbrace{\sum_{s=1}^S H_s x_s^*}_{=0}$$

Continues next slide... □

Lagrangian Relaxation

Proof.
Furthermore

$$\sum_{s=1}^{S} \pi_s \big(c^\top x_s^* + q_s^\top y_s^* \big) + \hat{\lambda} \underbrace{\sum_{s=1}^{S} H_s x_s^*}_{=0}$$

$$\geq \min_{x,y} \left\{ \sum_{s=1}^{s} \pi_s \big(c^\top x_s + q_s^\top y_s \big) + \hat{\lambda} \sum_{s=1}^{S} H_s x_s : (x_s, y_s) \in \mathcal{S}_s, s = 1, \ldots, S \right\}$$

$$= \min_{x,y} \left\{ \sum_{s=1}^{s} L_s(x_s, y_s, \hat{\lambda}) : (x_s, y_s) \in \mathcal{S}_s, s = 1, \ldots, S \right\} = D(\hat{\lambda})$$

□

Back

We can close the gap!

Proof.
Take $\hat{\lambda}$, solve $D(\hat{\lambda})$ and assume $(\hat{x}_s, \hat{y}_s)_{s=1}^S$ is feasible for the SP.

$$D(\hat{\lambda}) = \sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s\big) + \hat{\lambda} \underbrace{\sum_{s=1}^{S} H_s \hat{x}_s}_{=0} = \underbrace{\sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s\big)}_{\text{Objective of } (\hat{x}_s, \hat{y}_s)_{s=1}^S \text{ in SP}}$$

Continues next slide ... □

We can close the gap!

Proof.
On the other hand

$$D(\hat{\lambda}) = \sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s \big) + \hat{\lambda} \underbrace{\sum_{s=1}^{S} H_s \hat{x}_s}_{=0} \leq \max_{\lambda} D(\lambda) = z_{LD}$$

Thus

$$\underbrace{\sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s \big)}_{\text{Objective of } (\hat{x}_s, \hat{y}_s)_{s=1}^{S} \text{ in SP}} \leq z_{LD}$$

That is, z_{LD} is an upper bound the objective value of $(\hat{x}_s, \hat{y}_s)_{s=1}^{S}$.
Continues next slide ... □

We can close the gap!

Proof.

However, we know that z_{LD} is a lower bound.

Therefore

$$z_{LD} \leq \sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s\big) \leq z_{LD}$$

This holds only if

$$\sum_{s=1}^{S} \pi_s \big(c^\top \hat{x}_s + q_s^\top \hat{y}_s\big) = z_{LD}$$

This, $\hat{\lambda}$ is optimal for the dual and $(\hat{x}_s, \hat{y}_s)_{s=1}^S$ is optimal for the primal. □

Back

Proof optimality gap

$$D(\lambda) = \sum_{s=1}^{S} \min_{x_s, y_s} \left\{ \pi_s \big(c^\top x_s + q_s^\top y_s \big) + \lambda H_s x_s : (x_s, y_s) \in \mathcal{S}_s, s = 1, \dots, S \right\}$$

$$= \sum_{s=1}^{S} \min_{x_s, y_s} \left\{ \pi_s \big(c^\top x_s + q_s^\top y_s \big) + \lambda H_s x_s : (x_s, y_s) \in conv\mathcal{S}_s, s = 1, \dots, S \right\}$$

Continues next slide ...

Proof optimality gap

Therefore we can rewrite the dual as

$$
\begin{aligned}
z_{LD} &= \max_{\lambda} D(\lambda) \\
&= \max_{\lambda} \sum_{s=1}^{S} \min_{x_s, y_s} \left\{ \pi_s \big(c^\top x_s + q_s^\top y_s \big) + \lambda H_s x_s : (x_s, y_s) \in conv\mathcal{S}_s, s = 1, \ldots, S \right\}
\end{aligned}
$$

Continues next slide ...

Proof optimality gap

Therefore we can rewrite the dual as

$$
\begin{aligned}
z_{LD} &= \max_{\lambda} D(\lambda) \\
&= \max_{\lambda} \sum_{s=1}^{S} \min_{x_s, y_s} \left\{ \pi_s \big(c^\top x_s + q_s^\top y_s \big) + \lambda H_s x_s : (x_s, y_s) \in conv\mathcal{S}_s, s = 1, \ldots, S \right\}
\end{aligned}
$$

If $conv\mathcal{S}_s = \emptyset$ for some s, $z_{LD} = \infty$, (SP is infeasible).

Otherwise, assume $conv\mathcal{S}_s$ is bounded for all s and let (x_s^k, y_s^k) for $k \in \mathcal{K}_s$ be its extreme points. (Continues next slide ...)

Proof optimality gap

The optimum of each $D(\lambda)$ is attained at one of its extreme points...

$$D(\lambda) = \sum_{s=1}^{S} \min_{k \in \mathcal{K}_s} \left\{ \pi_s \left(c^\top x_s^k + q_s^\top y_s^k \right) + \lambda H_s x_s^k \right\}$$

and, in turn

$$z_{LD} = \max_{\lambda} \sum_{s=1}^{S} \min_{k \in \mathcal{K}_s} \left\{ \pi_s \left(c^\top x_s^k + q_s^\top y_s^k \right) + \lambda H_s x_s^k \right\}$$

Continues next slide ...

Proof optimality gap

The same problem can be rewritten as follows

$$z_{LD} = \max_{\lambda,\mu} \sum_{s=1}^{S} \mu_s$$
$$\mu_s \leq \pi_s\left(c^\top x_s^k + q_s^\top y_s^k\right) + \lambda H_s x_s^k \quad k \in \mathcal{K}_s, s = 1, \dots, S$$

by bringing all the decision variables on the left-hand-side ...

$$z_{LD} = \max_{\lambda,\mu} \sum_{s=1}^{S} \mu_s$$
$$\mu_s - H_s x_s^k \lambda \leq \pi_s\left(c^\top x_s^k + q_s^\top y_s^k\right) \qquad (\alpha_{ks}) \quad k \in \mathcal{K}_s, s = 1, \dots, S$$

Continues next slide ...

Proof optimality gap

Let us now take the dual of

$$z_{LD} = \max_{\lambda,\mu} \sum_{s=1}^{S} \mu_s$$

$$\mu_s - H_s x_s^k \lambda \leq \pi_s (c^\top x_s^k + q_s^\top y_s^k) \qquad (\alpha_{ks}) \quad k \in \mathcal{K}_s, s = 1, \ldots, S$$

$$z_{LD} = \min \sum_{s=1}^{S} \sum_{k \in \mathcal{K}_s} \pi_s (c^\top x_s^k + q_s^\top y_s^k) \alpha_{ks}$$

$$\sum_{k \in \mathcal{K}_s} \alpha_{ks} = 1 \qquad s = 1, \ldots, S$$

$$\sum_{s=1}^{S} \sum_{k \in \mathcal{K}_s} -H_s x_s^k \alpha_{ks} = 0$$

$$\alpha_{ks} \geq 0 \qquad k \in \mathcal{K}_s, s = 1, \ldots, S$$

Continues next slide ...

Proof optimality gap

The dual is selecting points in the convex hulls, provided that the points selected are non-anticipative, that is

$$
\begin{aligned}
z_{LD} = \min & \sum_{s=1}^{S} \sum_{k \in \mathcal{K}_s} \pi_s \left(c^\top x_s^k + q_s^\top y_s^k \right) \alpha_{ks} \\
& \sum_{k \in \mathcal{K}_s} \alpha_{ks} = 1 && s = 1, \ldots, S \\
& \sum_{s=1}^{S} \sum_{k \in \mathcal{K}_s} -H_s x_s^k \alpha_{ks} = 0 \\
& \alpha_{ks} \geq 0 && k \in \mathcal{K}_s, s = 1, \ldots, S
\end{aligned}
$$

corresponds to

$$
z_{LD} = \min \left\{ \sum_{s=1}^{S} \pi_s \left(c^\top x_s + q_s^\top y_s \right) : (x_s, y_s) \in conv \mathcal{S}_s, s = 1, \ldots, S, x_1 = \cdots = x_S \right\}
$$

This completes the proof Back

Proof concavity

$D(\lambda)$ is concave in λ.

Proof.
Take λ_1 and λ_2. You need to show that

$$\alpha D(\lambda_1) + (1-\alpha)D(\lambda_2) \leq D\big(\alpha\lambda_1 + (1-\alpha)\lambda_2\big)$$

with $\alpha \in [0, 1]$. □

Back